

# Optimal refrigerator

Armen E. Allahverdyan<sup>1</sup>, Karen Hovhannisyan<sup>1</sup>, Guenter Mahler<sup>2</sup>

<sup>1</sup>*Yerevan Physics Institute, Alikhanian Brothers Street 2, Yerevan 375036, Armenia,*

<sup>2</sup>*Institute of Theoretical Physics I, University of Stuttgart, Pfaffenwaldring 57, 70550 Stuttgart, Germany*

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We study a refrigerator model which consists of two  $n$ -level systems interacting via a pulsed external field. Each system couples to its own thermal bath at temperatures  $T_h$  and  $T_c$ , respectively ( $\theta \equiv T_c/T_h < 1$ ). The refrigerator functions in two steps: thermally isolated interaction between the systems driven by the external field and isothermal relaxation back to equilibrium. There is a complementarity between the power of heat transfer from the cold bath and the efficiency: the latter nullifies when the former is maximized and *vice versa*. A reasonable compromise is achieved by optimizing the product of the heat-power and efficiency over the Hamiltonian of the two system. The efficiency is then found to be bounded from below by  $\zeta_{CA} = \frac{1}{\sqrt{1-\theta}} - 1$  (an analogue of the Curzon-Ahlborn efficiency), besides being bound from above by the Carnot efficiency  $\zeta_C = \frac{1}{1-\theta} - 1$ . The lower bound is reached in the equilibrium limit  $\theta \rightarrow 1$ . The Carnot bound is reached (for a finite power and a finite amount of heat transferred per cycle) for  $\ln n \gg 1$ . If the above maximization is constrained by assuming homogeneous energy spectra for both systems, the efficiency is bounded from above by  $\zeta_{CA}$  and converges to it for  $n \gg 1$ .

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## I. INTRODUCTION

Thermodynamics studies principal limitations imposed on the performance of thermal machines, be they macroscopic heat engines or refrigerators [1–3], or small devices in nanophysics [4] and biology [5]. Taking as an example a refrigerator driven by a source of work, we recall three basic characteristics applicable to any thermal machine:

- Heat  $Q_c$  transferred per cycle of operation from a cold body at temperature  $T_c$  to a hot body at temperature  $T_h$  ( $T_h > T_c$ ).
- Power, which is the transferred heat  $Q_c$  divided over the cycle duration  $\tau$ .
- Efficiency (or performance coefficient)  $\zeta = Q_c/W$ , which quantifies the useful output  $Q_c$  over the work  $W$  consumed from the work-source for making the cycle. Note that work-consumption is obligatory, since the heat is transferred from cold to hot, i.e., against its natural gradient.

The second law imposes the Carnot bound

$$\zeta \leq \zeta_C = T_c/(T_h - T_c)$$

on the efficiency of refrigeration [2]. Within the usual thermodynamics the Carnot bound (both for heat-engines and refrigerators) is reached only for a reversible, i.e., an infinitely slow process, which means it is reached at zero power [2, 3]. The practical value of the Carnot bound is frequently questioned on this ground.

The drawback of zero power is partially cured within finite-time thermodynamics (FTT), which is still based on the quasi-equilibrium concepts [6]. For heat-engines FTT gives an *upper* bound  $\eta_{\text{opt}} \leq \eta_{CA} \equiv 1 - \sqrt{T_c/T_h}$ ,

where  $\eta_{\text{opt}}$  is the efficiency at the maximal power of work-extraction [7]. Naturally,  $\eta_{CA}$  is smaller than the Carnot upper bound  $1 - T_c/T_h$  for heat-engines.

Heat engines have recently been studied within microscopic theories, where one is easily able to go beyond the quasi-equilibrium regime [8–15]. For certain classes of heat-engines the CA efficiency is a *lower* bound for the efficiency at the maximal power of work [8–10]. This bound is reached at the quasi-equilibrium situation  $T_h \rightarrow T_c$  in agreement with the finding of FTT. The result is consistent with other studies [11, 12].

The interest in small-scale refrigerators is triggered by the importance of cooling processes for functioning of small devices and for displaying quantum features of matter [4, 15–20]. In particular, the theory of these refrigerators can provide answers to several basic questions such as how the third law limits the performance of a cooling machine at low temperatures [16], and how small are the temperatures reachable within a finite working time and under a reasonable amount of resource. Naturally, the small-scale refrigerators should also operate at a finite power. Note that the mirror symmetry between heat-engines and refrigerators, which is well-known for the zero-power case [2], does not hold more generally [21].

The present situation with finite-power refrigerators is somewhat unclear [21–24]. Here maximizing the power of cooling does not lead to reasonable results, since there is an additional complementarity (not present for heat engines) [17, 18, 21, 22]: when maximizing the heat-transfer power one simultaneously minimizes the efficiency to zero, and *vice versa*.

Here we intend to study optimal regimes of finite-power refrigeration via a model which can be optimized over almost all of its parameters. The model represents a junction immersed between two thermal baths at different temperatures and driven by an external work-source.

This type of models is frequently studied for modelling heat transport; see, e.g., [4, 19, 25]. Our model is quantum, but it admits a classical interpretation, because all the involved density matrices will be diagonal [in the energy representation] at initial and final moments of studied processes<sup>1</sup>.

This paper is organized as follows. The model is introduced in section II. Here we also show that the efficiency of the model is bounded by the Carnot value, and provide a general discussion of the refrigeration power. We confirm the heat-power-efficiency complementarity in section III and conclude that the most meaningful way of optimizing its functioning is to maximize the product of efficiency and the heat power. The optimization procedure is reported in section IV. We discuss the quasi-equilibrium limit of our model in section V. There we show that there is a *lower* bound  $\zeta_{CA} = -1 + 1/\sqrt{1-\theta}$  ( $\theta \equiv T_c/T_h$ ) for the efficiency, in addition to the upper Carnot bound  $\zeta_C = \frac{\theta}{1-\theta}$ . The same expression  $\zeta_{CA}$  was obtained within finite-time thermodynamics as an *upper* bound when optimizing the product of heat-power and efficiency or the ratio of the efficiency over the cycle time [21, 22]. Section VI discusses the attainability of the Carnot efficiency at a finite power. Entropy production inherent in the functioning of the model refrigerator is studied in section VII, while in section VIII we outline consequences of constraining features of the model to the quasi-classical domain. This constraint allows to reproduce the prediction of FTT on the upper bound of  $\zeta_{CA}$ . We summarize in section IX. Some technical questions are relegated to Appendix.

## II. THE MODEL

Consider two quantum systems **H** and **C** with Hamiltonians  $H_{\mathbf{H}}$  and  $H_{\mathbf{C}}$ , respectively. Each system has  $n$  energy levels. **H** and **C** constitute the working medium of our refrigerator; see Fig. 1.

Initially, **H** and **C** do not interact and are in equilibrium at temperatures  $T_h = 1/\beta_h > T_c = 1/\beta_c$  [we set  $k_B = 1$ ]:

$$\rho = e^{-\beta_h H_{\mathbf{H}}} / \text{tr}[e^{-\beta_h H_{\mathbf{H}}}], \quad \sigma = e^{-\beta_c H_{\mathbf{C}}} / \text{tr}[e^{-\beta_c H_{\mathbf{C}}}], \quad (1)$$

where  $\rho$  and  $\sigma$  are the initial Gibbsian density matrices

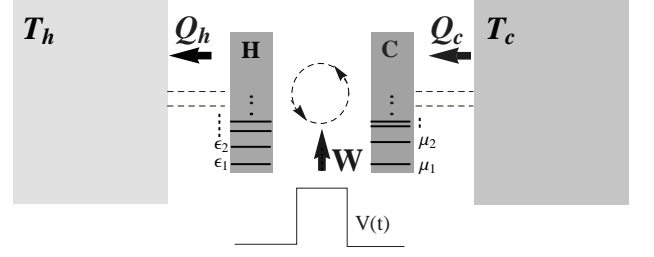


FIG. 1: The refrigerator model. Two systems **H** and **C** operate between two baths at temperatures  $T_c < T_h$  and are driven by an external potential  $V(t)$ .  $W$  and  $Q_c$  and  $Q_h$  are, respectively, the work put into the overall system and the heats transferred from the cold bath and to the hot bath.

of **H** and **C**, respectively. We write

$$\rho = \text{diag}[r_n, \dots, r_1], \quad \sigma = \text{diag}[s_n, \dots, s_1], \quad (2)$$

$$H_{\mathbf{H}} = \text{diag}[\varepsilon_n, \dots, \varepsilon_1 = 0], \quad H_{\mathbf{C}} = \text{diag}[\mu_n, \dots, \mu_1 = 0], \quad (3)$$

where  $\text{diag}[a, \dots, b]$  is a diagonal matrix with entries  $(a, \dots, b)$ , and where without loss of generality we have nullified the lowest energy level of both **H** and **C**. Thus the overall initial density matrix is

$$\Omega_{\text{in}} = \rho \otimes \sigma, \quad (4)$$

and the initial Hamiltonian  $H_{\mathbf{H}} \otimes 1 + 1 \otimes H_{\mathbf{C}}$ .

The goal of any refrigerator is to transfer heat from the cooler bath to the hotter one at the expense of consuming work from an external source. The present refrigerator model functions in two steps: thermally isolated work-consumption and isothermal relaxation; see Fig. 1. Let us describe these steps in detail.

**1.** **H** and **C** interact with each other and with the external sources of work. The overall interaction is described via a time-dependent potential  $V(t, \delta)$  in the total Hamiltonian

$$H(t, \delta) = H_{\mathbf{H}} \otimes 1 + 1 \otimes H_{\mathbf{C}} + V(t, \delta) \quad (5)$$

of **H** + **C**. The interaction process is thermally isolated:  $V(t, \delta)$  is non-zero only in a short time-window  $0 \leq t \leq \delta$  and is so large there that the influence of all other couplings [e.g., couplings to the baths] can be neglected [pulsed regime]. The time-dependent potential  $V(t, \delta)$  may explicitly depend on the coupling time  $\delta$ .

Thus the dynamics of **H** + **C** is unitary for  $0 \leq t \leq \delta$ :

$$\Omega_f \equiv \Omega(\delta) = \mathcal{U} \Omega_i \mathcal{U}^\dagger, \quad \mathcal{U} = \mathcal{T} e^{-\frac{i}{\hbar} \int_0^\delta ds V(s, \delta)}, \quad (6)$$

where  $\Omega_i = \Omega(0) = \rho \otimes \sigma$  is the initial state defined in (1),  $\Omega_f$  is the final density matrix,  $\mathcal{U}$  is the unitary evolution operator, and where  $\mathcal{T}$  is the time-ordering operator. The work put into **H** + **C** reads [1, 2]

$$W = E_f - E_i = \text{tr}[(H_{\mathbf{H}} \otimes 1 + 1 \otimes H_{\mathbf{C}})(\Omega_f - \Omega_i)], \quad (7)$$

<sup>1</sup> This aspect is similar to the Ising model. This is a model for quantum-mechanical spin- $\frac{1}{2}$ , but it can be given a classical interpretation via an overdamped particle moving in a asymmetric double-well potential. If the transversal components of the quantum spin are excited, this analogy breaks down. However, it still holds for the spin-flip process, where the transversal components are absent both initially and finally. In fact, the dynamics of the Ising model is introduced via such spin-flip processes, and this dynamics admits a classical interpretation.

where  $E_f$  and  $E_i$  are initial and final energies of  $\mathbf{H} + \mathbf{C}$ .

**2.** Once the overall system  $\mathbf{H} + \mathbf{C}$  arrives at the final state  $\Omega_{\text{fin}}$ ,  $V(t, \delta)$  is switched off, and  $\mathbf{H}$  and  $\mathbf{C}$  (within some relaxation time) return back to their initial states (1) under influence of the hot and cold thermal baths, respectively. Thus the cycle is complete and can be repeated again. Because the energy is conserved during the relaxation, the hot bath gets an amount of heat  $Q_h$ , while the cold bath gives up the amount of heat  $Q_c$ :

$$Q_h = \text{tr}(H_{\mathbf{H}}[\text{tr}_{\mathbf{C}}\Omega_f - \rho]), \quad Q_c = \text{tr}(H_{\mathbf{C}}[\sigma - \text{tr}_{\mathbf{H}}\Omega_f]), \quad (8)$$

where  $\text{tr}_{\mathbf{H}}$  and  $\text{tr}_{\mathbf{C}}$  are the partial traces. Eq. (1) and the unitarity of  $\mathcal{U}$  lead to

$$\beta_h Q_h - \beta_c Q_c = S(\Omega_f || \Omega_i) \equiv \text{tr}[\Omega_f \ln \Omega_f - \Omega_f \ln \Omega_i], \quad (9)$$

where  $S(\Omega_f || \Omega_i) \geq 0$  is the relative entropy, which employed in deriving thermodynamic bounds since [1, 35].

$S(\Omega_f || \Omega_i)$  nullifies if and only if  $\Omega_f = \Omega_i$ ; otherwise it is positive. Eq. (9) is the Clausius inequality, with  $S(\Omega_f || \Omega_i) \geq 0$  quantifying the entropy production. This point will be re-addressed and confirmed in section VII.

Eqs. (7–9) and the energy conservation  $Q_h = W + Q_c$  imply

$$(\beta_c - \beta_h)Q_c \leq \beta_h W, \quad (10)$$

meaning that in the refrigeration regime we have  $Q_c > 0$  and thus  $W > 0$ . Thus within the step **1** the work source transfers some energy from  $\mathbf{C}$  to  $\mathbf{H}$ , while in the step **2**,  $\mathbf{C}$  recovers this energy from the cold bath thereby cooling it and closing the cycle.

Eq. (9) leads to the Carnot bound for the efficiency  $\zeta$  [we denote  $\theta \equiv T_c/T_h < 1$ ]

$$\zeta \equiv \frac{Q_c}{W} = \frac{\theta}{1 - \theta} - \frac{S(\Omega_f || \Omega_i)}{(\beta_c - \beta_h)W} \leq \frac{\theta}{1 - \theta} \equiv \zeta_{\text{C}}. \quad (11)$$

We note from (11) that the deviation from the Carnot bound is controlled by the ratio of the entropy production  $S(\Omega_f || \Omega_i)$  to the work  $W$ .

We note in passing that all quantities introduced so far are meaningful also without the stage **2**. Then the problem reduces to cooling the initially equilibrium system  $\mathbf{C}$  with help of the work-source and the system  $\mathbf{H}$ . Both the work-source and  $\mathbf{H}$  are clearly necessary to achieve cooling<sup>2</sup>.  $Q_c$  quantifies the amount of cooling, while  $\zeta$  accounts for the relative effort of cooling.

<sup>2</sup> Indeed, if  $\mathbf{C}$  and the work-source form a closed system, no cooling is possible due to the Thomson's formulation of the second law [13] (cyclic processes cannot lead to work-extraction). If  $\mathbf{H}$  and  $\mathbf{C}$  form a closed system, then  $W = 0$  and no cooling is possible due to (10).

## A. Power

Recall that the power of refrigeration  $Q_c/\tau$  is defined as the ratio of the transferred heat  $Q_c$  to the cycle duration  $\tau$ . For our model  $\tau$  is limited mainly by the duration of the second stage, i.e.,  $\tau$  should be larger than the relaxation time  $\tau_{\text{rel}}$ , which depends on the concrete physics of the system-bath coupling.

Though some aspects of the following discussion are rather general, it will be useful to have in mind a concrete relaxation scenario. Consider the collisional relaxation scenario, where the target system interacts with independent bath particles via successive collisions; see [35, 36] and Appendix A. For our purpose the target system is  $\mathbf{H}$  or  $\mathbf{C}$  that interact with, respectively, the hot and cold bath. Each collision lasts a time  $\tau_{\text{col}}$ , which is much smaller than the characteristic time  $\tau_{\text{btw}}$  between two collisions. The interaction Hamiltonian between the target system and a bath particle is conserved, so that no work is done in switching the system-bath interaction on and off; see [35, 36] and Appendix A 1.

The relaxation process is typically (but not always) exponential with the characteristic relaxation time depending on the collisional interaction; see Appendix A 2. This time can be much smaller than any characteristic time of  $\mathbf{H}$  or  $\mathbf{C}$ . Since the two baths act on  $\mathbf{H}$  and  $\mathbf{C}$  independently, the overall relaxation process drives  $\mathbf{H} + \mathbf{C}$  to the initial state (1, 4).

If the interaction time  $\delta$  of  $V(t, \delta)$  [see (6)] is also much smaller than  $\tau_{\text{btw}}$ , one realizes a thermally-isolated process, because the overlap between the pulse and a collision can be neglected<sup>3</sup>.

To achieve a cyclic process within the exponential relaxation with the relaxation time  $\tau_{\text{rel}}$ , the cycle time  $\tau$  should be larger than  $\tau_{\text{rel}}$ . For each cycle the deviation of the post-relaxation state from the exact equilibrium state (1, 4) will be of order  $e^{-\tau/\tau_{\text{rel}}}$ . Thus if the ratio  $\tau/\tau_{\text{rel}}$  is simply large, but finite, one can perform roughly  $\sim e^{\tau/\tau_{\text{rel}}} \gg 1$  number of cycles at a finite power, before deviations from cyclicity would accumulate and the refrigerator will need resetting.

Though, as we stressed above, the relaxation process is

<sup>3</sup> Analogous conclusion on the irrelevance of the system-bath interaction during the pulse action is obtained when this interaction is always on, but its magnitude is small [weak-coupling]. Now the relaxation time is much larger than the internal characteristic time of  $\mathbf{H}$  and  $\mathbf{C}$ . Because the system-bath interaction is always on, there will be a contribution in the work (7) coming from the system-bath interaction Hamiltonian [37]. This contribution arises even when the conditions for the pulsed regime hold [37]. However, within the weak coupling assumption this additional contribution is proportional to the square of the system-bath interaction constant and can be neglected [37]. We stress that this additional contribution does not arise within the collisional relaxation scenario, because the pulse and collisions are well-separated in time.

normally exponential<sup>4</sup>, there are also situations within the collision relaxation scenario, where the system settles in the equilibrium state after just one inter-collision time  $\tau_{\text{btw}}$ ; see section VIA and Appendix A2 for details. The above limitations on the number of cycles does not apply to this relaxation scenario.

### 1. Comparing with the power of the Carnot cycle

The above situation does differ from the power consideration of usual (reversible) thermodynamic cycles, e.g., the Carnot cycle [2, 3, 26, 27]. There the external fields driving the working medium through various stages have to be much slower than the relaxation to the momentary equilibrium. The latter means that the working medium is described by its equilibrium Gibbs distribution with time-dependent parameters. The condition of momentary equilibrium for the working medium is necessary for the Carnot cycle to reach the Carnot efficiency [2, 3].

The precise meaning of the external fields being slow is important here. If  $\tau_F$  is the characteristic time of the fields, then the deviations from the momentary equilibrium are of order  $\mathcal{O}[\frac{\tau_{\text{rel}}}{\tau_F}]$  [3, 26, 27]. This fact is rather general and does not depend on details of the system and of the studied process, e.g., it does not depend whether the process is thermally isolated or adiabatic<sup>5</sup>. In particular, it is this deviation of the state from the momentary equilibrium that brings in the entropy production (or work dissipation) of order of  $\mathcal{O}[\left(\frac{\tau_{\text{rel}}}{\tau_F}\right)^2]$  [3, 26, 27].

Thus performing the reversible Carnot cycle with (approximately) the Carnot efficiency means keeping the ratio  $\frac{\tau_{\text{rel}}}{\tau_F}$  very small.

Now there are two basic differences between the Carnot cycle and our situation:

- In our case we do not require the working medium to be close to its momentary equilibrium state during the whole process. It suffices that the medium gets enough time to relax to its final equilibrium.

<sup>4</sup> More generally, the relaxation need not be exponential, but it still can be such that although the difference between the system density-matrix at time  $t$  and the corresponding Gibbsian density matrix goes to zero for  $t \rightarrow \infty$ , this difference does not turn to zero after any finite  $t$ . One of referees of this paper pointed out to us that *i*) the latter feature holds for a rather general class of relaxation processes taking place under a constant Hamiltonian; *ii*) it is rooted in the Kubo-Martin-Schwinger (KMS) condition [38] for correlation function evaluated over an equilibrium state; see [3] for a heuristic version of this argument; *iii*) the collisional relaxation is different in this respect, because the Hamiltonian is not constant. As we stress in Appendix A, a general point of the collisional relaxation is that no work is involved in this time-dependence.

<sup>5</sup> If a slow thermally isolated process is performed on a finite system, there are additional limitations in achieving the momentary equilibrium; see [26, 28] for more details. These limitations are however not essential for the present argument.

- A small, but finite  $\frac{\tau_{\text{rel}}}{\tau_F}$  for the Carnot cycle situation means that deviations from the momentary equilibrium are visible already within one cycle. In contrast, a small, but finite  $\frac{\tau_{\text{rel}}}{\tau}$  for our situation means that we can perform an exponentially large number of cycles before deviations from the cyclicity will be sizable. Here is a numerical example. Assume that  $\frac{\tau_{\text{rel}}}{\tau} = \frac{\tau_{\text{rel}}}{\tau_F} = 1/20$ . For the standard Carnot cycle already within one cycle the deviation from the momentary equilibrium will amount to 0.05. In our situation the same amount  $e^{-3} = 0.0498$  of deviation from the cyclicity will come after  $e^{17} = 2.4 \times 10^7$  cycles. This is a large number, especially taking into account that no realistic machine is supposed to work indefinitely long. Such machines do need resetting or repairing. The point is that our machine can perform *many* cycles at a finite power before any resetting is necessary.

## III. COMPLEMENTARITY BETWEEN THE TRANSFERRED HEAT AND EFFICIENCY

We now proceed to optimizing the functioning of the refrigerator over the three sets of available parameters: the energy spacings  $\{\varepsilon_k\}_{k=2}^n$ ,  $\{\mu_k\}_{k=2}^n$ , and the unitary operators  $\mathcal{U}$ . It should be evident from (5, 1) that optimizing over these parameters is equivalent to optimizing over the full time-dependent Hamiltonian  $H(t, \delta)$  of  $\mathbf{H} + \mathbf{C}$ . We stress in this context that no limitations on the magnitude of  $V(t, \delta)$  are imposed. This means that the unitary operator can in principle be generated in an arbitrary short coupling time  $\delta$ .

We start by maximizing the transferred heat  $Q_c = \text{tr}(H_C[\sigma - \text{tr}_{\mathbf{H}}\Omega_f])$ , which is the main characteristics of the refrigerator. Since  $\text{tr}[H_C\sigma]$  depends only on  $\{\mu_k\}_{k=2}^n$ , we choose  $\{\varepsilon_k\}_{k=2}^n$  and  $V(t)$  so that the final energy  $\text{tr}[H_C\Omega_f]$  attains its minimal value zero. Then we maximize  $\text{tr}[H_C\sigma]$  over  $\{\mu_k\}_{k=2}^n$ . Note from (2)

$$1 \otimes H_C = \text{diag}[\mu_1, \dots, \mu_1, \dots, \mu_n, \dots, \mu_n], \\ \Omega_i = \rho \otimes \sigma = \text{diag}[s_1 r_1, \dots, s_1 r_n, \dots, s_n r_1, \dots, s_n r_n].$$

It is clear that  $\text{tr}[H_C\Omega_f] = \text{tr}[H_C\mathcal{U}\Omega_i\mathcal{U}^\dagger]$  goes to zero when, e.g.,  $r_2 = \dots = r_n \rightarrow 0$  ( $\varepsilon \equiv \varepsilon_2 = \dots = \varepsilon_n \rightarrow \infty$ ), while  $\mathcal{U}$  amounts to the SWAP operation  $\mathcal{U}\rho \otimes \sigma\mathcal{U}^\dagger = \sigma \otimes \rho$ . It is checked by a direct inspection that the maximization of the initial energy  $\text{tr}[H_C\sigma]$  over  $\{\mu_k\}_{k=2}^n$  produces the same structure of  $n-1$  times degenerate upper energy levels  $\mu \equiv \mu_2 = \dots = \mu_n$ . Denoting

$$v \equiv s_2 = \dots = s_n = e^{-\beta_c \mu}, \quad u \equiv r_2 = \dots = r_n = e^{-\beta_h \varepsilon}, \quad (12)$$

we obtain for  $Q_c$

$$Q_c = T_c \ln \left[ \frac{1}{v} \right] \frac{(v-u)(n-1)}{[1+(n-1)v][1+(n-1)u]}, \quad (13)$$

where according to the above discussion,  $Q_c$  is maximized for  $u \rightarrow 0$ , and where  $v$  is to be found from maximizing  $Q_c|_{u \rightarrow 0}$  in (13) over  $v$ , i.e.,  $v$  is determined via

$$1 + (n-1)v + \ln v = 0. \quad (14)$$

For the efficiency we get for the present situation ( $\mathbf{H}$  and  $\mathbf{C}$  have  $n-1$  times degenerate upper levels, while  $\mathcal{U}$  amounts to the SWAP operation):

$$\zeta = \frac{Q_c}{W} = \frac{\mu}{\varepsilon - \mu} = \frac{\theta \ln[\frac{1}{v}]}{\ln[\frac{1}{u}] - \theta \ln[\frac{1}{v}]}. \quad (15)$$

The maximization of  $Q_c$  led us to  $u \rightarrow 0$ , which then means that  $\zeta$  in (15) goes to zero.

Thus  $\mathbf{C}$  can be cooled down to its ground state ( $\text{tr}[H_{\mathbf{C}}\Omega_f] \rightarrow 0$ ), but at a vanishing efficiency, i.e., at expense of an infinite work. To make this result consistent with the classic message of the third law [33], we should slightly adjust the latter: one cannot reach the zero temperature [of an initially equilibrium system] in a finite time *and* with finite resources [infinite work is not a finite resource]. At any rate, one should note that the classic formulation of the third law motivates its operational statement using exclusively equilibrium concepts. Modern perspectives on the third law are discussed in [16, 18, 20, 29, 30].

Note that the efficiency  $\zeta$  in (15) reaches its maximal Carnot value  $\theta/(1-\theta)$  for

$$u = v, \quad (16)$$

which nullifies the transferred heat  $Q_c$ ; see (13).

Now we have to show that  $Q_c$  tends to zero upon maximizing  $\zeta$  over *all* free parameters  $\{\varepsilon_k\}_{k=2}^n$ ,  $\{\mu_k\}_{k=2}^n$  and  $\mathcal{U}$ . Denoting  $\{|i_{\mathbf{H}}\rangle\}_{k=1}^n$  and  $\{|i_{\mathbf{C}}\rangle\}_{k=1}^n$  for the eigenvectors of  $H_{\mathbf{H}}$  and  $H_{\mathbf{C}}$ , respectively, we note from (7, 8) that  $W$  and  $Q_c$  feel  $\mathcal{U}$  only via the matrix

$$C_{ij|kl} = |\langle i_{\mathbf{H}} | j_{\mathbf{C}} | \mathcal{U} | k_{\mathbf{H}} | l_{\mathbf{C}} \rangle|^2. \quad (17)$$

This matrix is double-stochastic [34]:

$$\sum_{ij} C_{ij|kl} = \sum_{kl} C_{ij|kl} = 1. \quad (18)$$

Conversely, for any double-stochastic matrix  $C_{ij|kl}$  there is some unitary matrix  $U$  with matrix elements  $U_{ij|kl}$ , so that  $C_{ij|kl} = |U_{ij|kl}|^2$  [34]. Thus, when maximizing various functions of  $W$  and  $Q_c$  over the unitary  $\mathcal{U}$ , we can directly maximize over the  $(n^2 - 1)^2$  independent elements of  $n^2 \times n^2$  double stochastic matrix  $C_{ij|kl}$ .

We did not find an analytic way of carrying out the complete maximization of  $\zeta$  over all free parameters. Thus we had to rely on numerical recipes of Mathematica 7, which for  $n = 1, \dots, 5$  confirmed that  $Q_c$  nullifies whenever  $\zeta$  reaches (along any path) its maximal Carnot value. We believe this holds for an arbitrary  $n$ , though we lack any rigorous proof of this assertion.

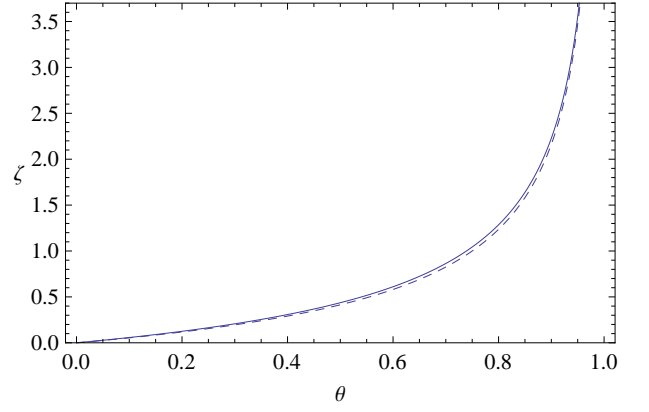


FIG. 2: Solid line: efficiency  $\zeta$  of the optimized refrigerator versus the temperature ratio  $\theta = T_c/T_h$  for  $n = 3$ ; see (15). In the scale of this figure  $\zeta(n=2)$  and  $\zeta(n=3)$  are almost indistinguishable. Dashed line: the lower bound  $\frac{1}{1-\theta} - 1$ .

#### IV. MAXIMIZING THE PRODUCT OF THE TRANSFERRED HEAT AND EFFICIENCY.

We saw above that neither  $Q_c$  nor  $\zeta$  are good target quantities for determining an optimal regime of refrigeration. But

$$\chi \equiv Q_c \zeta, \quad (19)$$

is such a target quantity, as will be seen shortly. This is the most natural choice for our setup. This choice was also employed in [21]. Refs. [17, 18, 24] report on different approaches to defining refrigeration regimes.

The numerical maximization of  $\chi = \zeta Q_c$  over  $\{\varepsilon_k\}_{k=2}^n$ ,  $\{\mu_k\}_{k=2}^n$  and  $\mathcal{U}$  has been carried out for  $n = 1, \dots, 5$  along the lines discussed around (17, 18). It produced the same structure: both  $\mathbf{H}$  and  $\mathbf{C}$  have  $n-1$  times degenerate upper levels, see (12), and the optimal  $\mathcal{U}$  again corresponds to SWAP operation <sup>6</sup>:

$$\Omega_i = \rho \otimes \sigma, \quad \Omega_f = \sigma \otimes \rho. \quad (20)$$

Recalling the expression (5) for the total Hamiltonian  $H(t, \delta)$ , we can state this result as follows: there exist a coupling potential  $V(t, \delta)$  that for a given coupling time  $\delta$  generates the unitary SWAP operation following to (6). This operation does not explicitly depend on  $\delta$ , because  $V(t, \delta)$  depends on  $\delta$ ; see also our discussion in the beginning of section III. Note that both the initial and final states in (20) are diagonal in the energy representation.

<sup>6</sup> Let us recall how the SWAP is defined via a pure-state base. Let  $\{|k\rangle_1\}_{k=1}^n$  be an orthonormal base in the Hilbert where  $\rho$  lives. Let also  $\{|k\rangle_2\}_{k=1}^n$  be an orthonormal base in the Hilbert space where  $\sigma$  lives. Any unitary operator acting on the composite Hilbert space can be defined with respect to the orthonormal base  $|k\rangle_1 \otimes |l\rangle_2$ , where  $k, l = 1, \dots, n$ . Let us now define for all pairs  $k$  and  $l$ :  $U_{\text{SWAP}}|k\rangle_1 \otimes |l\rangle_2 = |l\rangle_1 \otimes |k\rangle_2$ .

Evidently, the intermediate state  $\Omega(t)$  for  $0 < t < \delta$  is not diagonal in this representation.

The efficiency  $\zeta$  and the transferred heat  $Q_c$  are given by, respectively, (13) and (15), where instead of  $u$  and  $v$  we should substitute  $\bar{u}$  and  $\bar{v}$ , respectively. The latter two quantities are obtained from maximizing  $\chi = \zeta Q_c$ ,

$$\chi(\bar{u}, \bar{v}) = \frac{T_c \theta (n-1) (\bar{v} - \bar{u}) \ln^2 \frac{1}{\bar{v}}}{[\ln \frac{1}{\bar{u}} - \theta \ln \frac{1}{\bar{v}}][1 + (n-1)\bar{u}][1 + (n-1)\bar{v}]}, \quad (21)$$

where  $\bar{u}$  and  $\bar{v}$  are found from maximizing  $\chi(u, v)$  via  $\partial_u \chi = \partial_v \chi = 0$ . Note that  $\bar{u}$  and  $\bar{v}$  depend on  $\theta = T_c/T_h$ . The efficiency  $\zeta$  and the transferred heat  $Q_c$  are given, respectively, by (15) and (13) with  $u \rightarrow \bar{u}$  and  $v \rightarrow \bar{v}$ .

Though we have numerically checked these results for  $n \leq 5$  only, we again trust that they hold for an arbitrary  $n$  (one can, of course, always consider the above structure of energy spacings and  $\mathcal{U}$  as a useful ansatz).

SWAP is one of the basic gates of quantum information processing [31]; see [14] for an interesting discussion on the computational power of thermodynamic processes. SWAP is sometimes realized as a composition of more elementary unitary operations, but its direct realizations in realistic systems also attracted attention; see, e.g., [32] for a direct implementation of SWAP in quantum optics. Note that for implementing the SWAP as in (20) the external agent need not have any information on the actual density matrices  $\rho$  and  $\sigma$ .

### A. Effective temperatures

Since the state  $\Omega_f$  of  $\mathbf{H} + \mathbf{C}$  after the action of  $V(t)$  is  $\sigma \otimes \rho$ , and because in the optimal regime the upper level for both  $\mathbf{H}$  and  $\mathbf{C}$  is  $n-1$  times degenerate, one can introduce non-equilibrium temperatures  $T'_h$  and  $T'_c$  for respectively  $\mathbf{H}$  and  $\mathbf{C}$  via [note (1)]

$$\sigma = e^{-\beta'_h H_{\mathbf{H}}} / \text{tr}[e^{-\beta'_h H_{\mathbf{H}}}], \quad \rho = e^{-\beta'_c H_{\mathbf{C}}} / \text{tr}[e^{-\beta'_c H_{\mathbf{C}}}], \quad (22)$$

where we recall that  $\sigma$  ( $\rho$ ) is the state of  $\mathbf{H}$  ( $\mathbf{C}$ ) after applying the pulse. Using (12) we deduce

$$T'_h = T_h \frac{\ln \frac{1}{\bar{u}}}{\ln \frac{1}{\bar{v}}}, \quad T'_c = T_c \frac{\ln \frac{1}{\bar{v}}}{\ln \frac{1}{\bar{u}}}, \quad (23)$$

where  $\bar{v} = e^{-\beta_c \bar{\mu}}$  and  $\bar{u} = e^{-\beta_h \bar{\epsilon}}$ ; see (12). This implies

$$T_c T_h = T'_c T'_h. \quad (24)$$

As expected, the refrigeration condition  $\bar{v} > \bar{u}$ , see (13, 21), is equivalent to

$$T'_c < T_c < T_h < T'_h, \quad (25)$$

i.e., after the pulse the cold system gets colder, while the hot system gets hotter. Note that the existence of temperatures  $T'_c$  and  $T'_h$  was not imposed, they emerged

out of optimization. In terms of these temperatures the efficiency (15) is conveniently written as

$$\zeta = \frac{T_c}{T'_h - T_c} = \frac{T'_c}{T_h - T'_c}. \quad (26)$$

We eventually focus on two important limits: quasi-equilibrium  $\theta \rightarrow 1$ , and the regime  $\ln n \gg 1$ .

### V. QUASI-EQUILIBRIUM REGIME $\theta \rightarrow 1$ : A LOWER BOUND FOR THE EFFICIENCY

In this regime the temperatures  $T_h$  and  $T_c$  are nearly equal to each other:  $\theta \equiv T_c/T_h \rightarrow 1$ .

First we note that sharply at  $\theta = 1$ ,  $\chi$  reads

$$\chi(a)|_{\theta=1} = T_c \theta (n-1) [1 + (n-1)a]^{-2} a \ln^2 a, \quad (27)$$

where

$$\bar{u} = \bar{v} = a,$$

and where  $a$  is given by  $\partial_a \chi(a)|_{\theta=1} = 0$ :

$$[(n-1)a - 1] \ln a = 2[(n-1)a + 1]. \quad (28)$$

We now work out the optimal  $\bar{u}$  and  $\bar{v}$  for  $\theta \rightarrow 1$ . It can be seen from (21) that the proper expansion parameter for  $\theta \rightarrow 1$  is  $x \equiv \sqrt{1-\theta}$ . We write

$$\bar{u} = a + \sum_{k=1} a_k x^k, \quad \bar{v} = a + \sum_{k=1} (a_k + b_{k-1}) x^k. \quad (29)$$

We substitute (29) into  $\partial_u \chi = 0$  and  $\partial_v \chi = 0$  and expand them over  $x$ . Both expansions start from terms of order  $\mathcal{O}(x^0)$ . Now  $a_k$  and  $b_k$  are determined by equating to zero the  $\mathcal{O}(x^k)$  terms in  $\partial_u \chi = 0$  and  $\partial_v \chi = 0$ . Thus the  $\mathcal{O}(x^0)$  terms together with (28) define  $b_0$ :

$$b_0 = a \ln \frac{1}{a}, \quad (30)$$

which should be non-negative due to  $v > u$ . The  $\mathcal{O}(x^1)$  terms together with (28) and (30) define  $a_1$  and  $b_1$ :

$$a_1 = -\frac{a}{2} \ln \frac{1}{a}, \quad b_1 = -\frac{a[24 + \ln^2 a]}{48} \ln \frac{1}{a}, \quad (31)$$

and so on. Eqs. (29, 30, 31) imply for the efficiency at  $\theta \rightarrow 1$  ( $x = \sqrt{1-\theta}$ )

$$\zeta = \frac{1}{x} - 1 + \frac{\ln^2 a}{48} - \frac{[48 + \ln^2 a] \ln^2 a}{1536} x + \mathcal{O}(x^2). \quad (32)$$

Note that the expansion (32) does not apply for  $n \rightarrow \infty$ , since in this limit  $a(n-1) \simeq 1 - \frac{4}{\ln[n-1]}$ ; see (28). Thus, in the limit  $\theta \rightarrow 1$ ,  $Q_c$  scales as  $\propto \sqrt{1-\theta}$ ,

$$Q_c = \frac{a T_c [\ln a]^2 (n-1) \sqrt{1-\theta}}{[1 + (n-1)a]^2}, \quad (33)$$

while the consumed work is smaller and scales as  $1 - \theta$ .

Eq. (32) suggests that the maximization of  $\chi$  imposes a lower bound on the efficiency:

$$\zeta > \zeta_{\text{CA}} \equiv \frac{1}{\sqrt{1-\theta}} - 1. \quad (34)$$

This is numerically checked to be the case for all  $0 < \theta < 1$  and all  $n$ ; see also Fig. 2.

The expression of  $\zeta_{\text{CA}}$  was already obtained within finite-time thermodynamics—but as an upper bound on the efficiency—and argued to be an analogue of the Curzon-Ahlborn efficiency for refrigerators [21, 22]. Section VIII explains that also within the present microscopic approach  $\zeta_{\text{CA}}$  can be an upper bound for  $\zeta$  provided that  $\chi$  is maximized under certain constraints.

Recalling (15), our discussion after (23–25) and (26), we can interpret the lower bound for the efficiency as a lower bound on the intermediate temperature  $T'_c$  of  $\mathbf{C}$ :

$$\frac{1 - \sqrt{1-\theta}}{\theta} < \frac{T'_c}{T_c} < 1, \quad (35)$$

i.e., the lowest temperature  $T'_c$  cannot be too low under optimal  $\chi$ . Compare this with the fact that under vanishing efficiency (that is for very large amount of the consumed work),  $T'_c$  can be arbitrary low; see our discussion after (15). Thus a well-defined lowest (per cycle) temperature emerged once we restricted the resource of cooling (the consumed work).

## VI. THE MANY-LEVEL REGIME: REACHING THE CARNOT LIMIT AT A FINITE POWER.

Now we turn to studying the regime

$$\ln(n-1) \gg 1. \quad (36)$$

First of all, let us introduce two new variables

$$\rho \equiv \bar{u}(n-1) \ln[n-1], \quad \xi \equiv \frac{\ln[n-1]}{\bar{v}(n-1)}, \quad (37)$$

denote

$$p \equiv \ln[n-1], \quad (38)$$

and rewrite  $\chi$  in (21) as

$$\frac{\chi(1-\theta)}{\theta T_c p} = \frac{(1 - \frac{\rho\xi}{p^2})(1 + \frac{1}{p} \ln[\frac{\xi}{p}])^2}{(1 + \frac{\rho}{p})(1 + \frac{\xi}{p})(1 + \frac{1}{(1-\theta)p} \ln[\frac{2^{\theta+1}}{\rho\xi^\theta}])}. \quad (39)$$

The expression in RHS of (39) is now to be optimized over  $\rho$  and  $\xi$ . We note that if these parameters stay finite in the limit  $p \equiv \ln[n-1] \gg 1$ , the value of  $\chi$  is read off directly:  $\frac{\chi(1-\theta)}{\theta T_c p} = 1$ . The finiteness of  $\rho$  and  $\xi$  in the limit  $p \equiv \ln[n-1] \gg 1$  is confirmed by expanding the RHS of (39) over the small parameter  $\frac{1}{p}$ , collecting terms

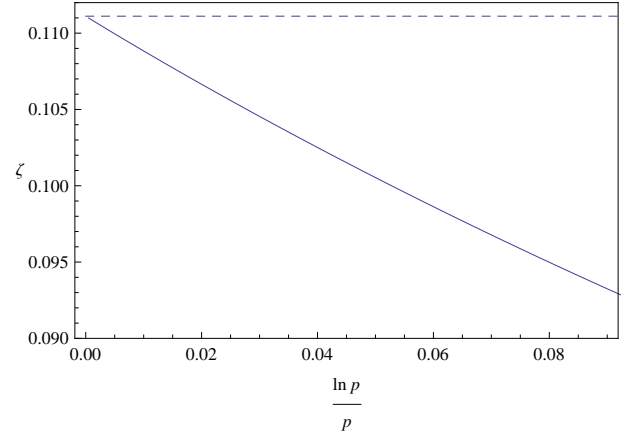


FIG. 3: Convergence of the efficiency  $\zeta$  (normal line) to the Carnot value  $\eta_C = 1/9$  (dashed line) as a function of  $\frac{\ln p}{p}$ , where  $p = \ln[n-1]$ ; see (36–39).

$\propto \mathcal{O}(\frac{1}{p})$ , differentiating them over  $\rho$  and  $\xi$ , and equating the resulting expressions to zero. This produces:

$$\rho = \frac{1}{1-\theta} + \mathcal{O}\left(\frac{1}{p}\right), \quad \xi = \frac{2-\theta}{1-\theta} + \mathcal{O}\left(\frac{1}{p}\right). \quad (40)$$

Substituting these into (15) and (21) we get

$$\zeta = \frac{\theta}{1-\theta} - \frac{2\theta}{(1-\theta)^2} \frac{\ln[p]}{p} + \mathcal{O}\left[\frac{1}{p^2}\right], \quad (41)$$

$$\frac{Q_c}{T_c} = p - \frac{3-\theta}{1-\theta} - \ln\left[\frac{1-\theta}{2-\theta} p\right] + \mathcal{O}\left[\frac{1}{p}\right]. \quad (42)$$

Note from (36, 38) that the dominant factor in the efficiency  $\zeta$  is the Carnot value  $\frac{\theta}{1-\theta}$ , while the subleading term is naturally negative; see (41). Likewise, the dominant factor in  $\frac{Q_c}{T_c}$  is  $p \equiv \ln[n-1]$ , while the subleading term is  $\mathcal{O}(1)$ . We also see that the limit  $\ln(n-1) \gg 1$  does not commute with the equilibrium limit  $\theta \rightarrow 1$ , since the corrections in (41, 42) diverge for  $\theta \rightarrow 1$ .

Thus in this regime  $p \equiv \ln(n-1) \gg 1$  the efficiency converges to the Carnot value; see Fig. 3.

Recalling (12) we see from (37, 40) that in the regime (36) the total occupation of the higher levels of  $\mathbf{H}$  is small, so that  $\mathbf{H}$  is predominantly in its ground state before applying the work-consuming external pulse. In contrast,  $\mathbf{C}$  is more probably in one of its excited states. These facts are expected, because  $\mathbf{C}$  has to give up some energy, while  $\mathbf{H}$  has to accept it.

We note that this regime resembles in several aspects the macroscopic regime  $N \gg 1$  of a  $N$ -particle system. Recall that for  $N \gg 1$  (weakly coupled) particles the number of energy levels scales as  $e^N$ , while energy scales as  $N$ . Now for the above situation (42) the transferred heat  $Q_c$  is (in the leading order) a product of the colder temperature  $T_c$  and the "number of particles"  $\ln(n-1)$ .

The effective temperatures  $T'_h$  and  $T'_c$  [see (23–25)] in



this limit are close to their initial values:

$$\frac{T'_h}{T_h} = 1 + \frac{1}{p} \ln \left[ \frac{(1-\theta)^2 p^2}{2-\theta} \right] + \mathcal{O} \left[ \frac{1}{p^2} \right], \quad (43)$$

$$\frac{T'_c}{T_c} = 1 - \frac{1}{p} \ln \left[ \frac{(1-\theta)^2 p^2}{2-\theta} \right] + \mathcal{O} \left[ \frac{1}{p^2} \right], \quad (44)$$

where we employed (37, 40). Though during the refrigeration process the systems **C** and **H** are able to process large amounts of work and heat ( $\propto \ln[n-1]$ ), their temperatures are not perturbed strongly.

### A. Finiteness of power

It is important to note that the asymptotic attainability (41) of the Carnot bound for  $\zeta$  is related to a finite transferred heat  $Q_c = T_c \ln[n-1]$ , but it also can be related to a finite power  $\frac{Q_c}{\tau}$ , where in our model the cycle time  $\tau$  basically coincides with the relaxation time; recall our discussion in section II A. This appears to be unexpected, because within the standard thermodynamic analysis the Carnot efficiency is reached by the Carnot cycle at a vanishing power [2]; see section II A 1 for a precise meaning of this statement. In any refrigerator model known to us—see, e.g., [19]—approaching the Carnot limit means nullifying the power. See also in this context our discussion around equation (16); various reasons preventing the approach to the Carnot efficiency for thermal machines (even for small machines working at zero power) are analysed in [26]. Now we supplement our discussion in section II A with more specific arguments.

We already stressed in section II A that within the second stage of the refrigerator functioning, where both **H** and **C** relax to equilibrium under influence of the corresponding thermal baths, the relaxation mechanism can be associated with the collisional system-bath interaction; see Appendix A for a detailed discussion of this mechanism. Here there are three characteristic times: the single collision duration time  $\tau_{\text{col}}$  is much smaller than the inter-collision time  $\tau_{\text{btw}}$ , while the relaxation of the system to its equilibrium state is governed by the time  $\tau_{\text{rel}}$ . The assumed condition  $\tau_{\text{col}} \ll \tau_{\text{btw}}$  allows implementing the thermally isolated work-consuming pulse, because if the pulse time is also much smaller than  $\tau_{\text{btw}}$ , the pulse does not overlap with collisions.

In Appendix A 2 we study the relaxation time of the system with  $n-1$  fold degenerate upper energy levels and non-degenerate lowest energy level. We also account for the limit  $\ln[n-1] \gg 1$ , where the Carnot efficiency is reached; see (41). It is shown that for such a system the relaxation time  $\tau_{\text{rel}}$  can—depending on the details of the thermal bath and its interaction with the system—range from few  $\tau_{\text{btw}}$ 's to a very long times  $\propto n\tau_{\text{btw}}$ . The former relaxation time means a finite power, while the latter time implies vanishing power  $\propto \frac{\ln n}{n}$  for  $\ln n \gg 1$ .

These two extreme cases are easy to describe without addressing the formalism of Appendices A 1 and A 2.

1. For simplicity let us focus on the relaxation of the system **H** that after the work-extracting pulse (20) is left in the state  $\sigma$ , and is now subjected to a stream of the bath particles (the situation with **C** is very similar). Recall that each bath particle before colliding with **H** is in the Gibbsian equilibrium state at the temperature  $T_h$ . Now assume that each bath particle also has  $n-1$  fold degenerate upper level, and one lowest energy level. Also, the non-zero energy spacing for the bath particles is equal to that of **H**. Then the relaxation of **H** is achieved just after one collision provided that the system-bath interaction [during this collision] amounts to a SWAP operation. Note that the characteristic time of this relaxation is  $\tau_{\text{btw}}$ , and that this is a non-exponential scenario of relaxation, because the system exactly settles into its equilibrium Gibbsian state after the first collision.

No work is done during collisional relaxation; see Appendix A. Indeed, under above assumptions on the energy levels of the bath particles, the SWAP operation commutes with the free Hamiltonian  $H_1 \otimes 1 + 1 \otimes H_{\mathbf{H}}$  (where  $H_1$  is the Hamiltonian of the given bath particle), which implies that the final energy of **H** plus the bath particle is equal to its initial value. Since each separate collision is a thermally isolated process, this means that no work is done; see (7).

2. If each collision is very weak and almost does not exchange heat with the system **H**, the relaxation time becomes very long. Intermediate cases are discussed in Appendix A 2. These intermediate cases are relevant, since the power of refrigeration is finite even for long relaxation times  $\propto \ln n$ . Indeed, we recall from (42) that  $Q_c = T_c \ln[n-1] + \mathcal{O}(1)$ .

### B. More realistic spectra still allowing to reach the Carnot bound

One can ask whether the convergence (41) to the Carnot bound is a unique feature of the spectra (12) in the limit  $\ln[n-1] \gg 1$ , or whether there are other situations that still allow  $\zeta \rightarrow \zeta_c$ . The answer is positive as we now intend to show. For the energy spectra (3) we postulate  $[k=1, \dots, n-1]$

$$\varepsilon_{k+1} = \varepsilon + (k-1)\delta, \quad \mu_{k+1} = \mu + (k-1)\delta, \quad (45)$$

where  $\delta > 0$  is a parameter. Next, we assume that the following six conditions hold

$$(n-1)\beta\delta \gg 1, \quad \beta_c\delta \ll 1, \quad \beta_h\delta \ll 1 \quad (46)$$

$$\bar{p} \equiv \ln[T_c/\delta] \gg 1, \quad (47)$$

$$u \equiv e^{-\beta_h\varepsilon} \simeq \delta/\bar{p}, \quad v \equiv e^{-\beta_c\mu} \simeq \bar{p}\delta. \quad (48)$$

Under conditions (46, 47, 48)—and *assuming* the SWAP operation for the pulse—we show below that the results analogous to (41, 42) hold,

$$\zeta = \frac{\theta}{1-\theta} + \mathcal{O} \left[ \frac{1}{\bar{p}} \right], \quad \frac{Q_c}{T_c} = \bar{p} \left( 1 + \mathcal{O} \left[ \frac{1}{\bar{p}} \right] \right). \quad (49)$$



where the role of a large parameter  $p = \ln[n-1]$  in (41, 42) is now played by  $\bar{p}$ <sup>7</sup>. Note that (46) and (47) still imply that  $\ln[n-1] \gg 1$ .

The spectra (45) under condition (46, 47, 48) correspond to a quasi-continuous part separated from the ground state by a gap. This type of spectrum avoids the strong degeneracy of (12), and is met in conventional superconductors below the transition temperature [39].

To derive (49) via (45–48) we note the following formulas for the partition sums  $Z_h \equiv \text{tr}[e^{-\beta_h H_{\mathbf{H}}}]$  and  $Z_c \equiv \text{tr}[e^{-\beta_c H_{\mathbf{C}}}]$  in (1), the heat  $Q_c$  and work  $W$ :

$$\begin{aligned} Z_h &= 1 + \frac{T_h u}{\delta}, & Z_c &= 1 + \frac{T_c v}{\delta}, \\ Q_c &= T_c \ln \left[ \frac{1}{v} \right] \left[ \frac{1}{Z_h} - \frac{1}{Z_c} \right] + \frac{T_c^2 v}{Z_c \delta} - \frac{T_h^2 u}{Z_h \delta}, \\ W &= T_h \left[ \ln \frac{1}{u} - \theta \ln \frac{1}{v} \right] \left[ \frac{1}{Z_h} - \frac{1}{Z_c} \right]. \end{aligned}$$

## VII. ENTROPY PRODUCTION

Entropy production is an important characteristics of thermal machines, because it quantifies the irreversibility of their functioning [1, 2]. For our refrigerator model, no entropy is produced during the first stage, which is thermally isolated from the baths. However, a finite amount of entropy is produced during the second, relaxational stage. The overall entropy production reads

$$S_i = S[\Omega_{\text{fin}} || \Omega_{\text{in}}],$$

and controls the deviation of efficiency from its maximal Carnot value; see (11). In the optimal conditions (20, 12), we get

$$S_i = S[\rho || \sigma] + S[\sigma || \rho] \quad (50)$$

$$= \frac{\ln[\bar{v}/\bar{u}] (\bar{v} - \bar{u})(n-1)}{[1 + (n-1)\bar{v}][1 + (n-1)\bar{u}]}, \quad (51)$$

where  $S[\sigma || \rho]$  and  $S[\rho || \sigma]$  are the entropies produced in, respectively, cold and hot bath. Indeed, consider the system  $\mathbf{C}$  that after the external field action is left in the state with density matrix  $\rho$  [see (20)], and now under influence of the thermal bath should return to its initial state  $\sigma \propto e^{-\beta_c H_{\mathbf{C}}}$ . Now

$$T_c S[\rho || \sigma] = \text{tr}[H_{\mathbf{C}} \rho] - T_c S[\rho] + T_c \ln \text{tr} e^{-\beta_c H_{\mathbf{C}}}, \quad (52)$$

is the difference between the non-equilibrium free energy  $\text{tr}[H_{\mathbf{C}} \rho] - T_c S[\rho]$  of  $\mathbf{C}$  in the state  $\rho$  [and in contact to a thermal bath at temperature  $T_c$ ] and the equilibrium free energy  $-T_c \ln \text{tr} e^{-\beta_c H_{\mathbf{C}}}$ . Simultaneously,  $T_c S[\rho || \sigma]$

in (52) is the maximal work that can be extracted from the system  $\mathbf{C}$  (in state  $\sigma$ ) in contact with the  $T_c$ -bath [8]. During relaxation this potential work is let to relax into the  $T_c$ -bath increasing its entropy by  $S[\rho || \sigma]$ . Likewise,  $S[\sigma || \rho]$  is the entropy production during the relaxation of the system  $\mathbf{H}$  in contact with the  $T_h$ -bath.

Now in the regime  $\ln[n-1] \gg 1$ ,  $S_i$  amounts to  $\ln[\bar{v}/\bar{u}] \simeq \ln(\ln[n-1])$ , see (51), while the consumed work  $W$  and the transferred heat  $Q_c$  scale as  $\ln[n-1]$ . In other words, the entropy production  $S_i$  is much smaller than both  $W$  and  $Q_c$ . This explains why for a large  $\ln[n-1]$  the Carnot efficiency is reached; see (11, 41).

In the equilibrium limit  $\theta \rightarrow 1$ ,  $S_i$  reads

$$S_i = \frac{a [\ln a]^2 (n-1) (1-\theta)}{[1 + (n-1)a]^2}, \quad (53)$$

where  $a$  is given by (28), and where in deriving (53) we employed (51) and asymptotic expansions presented after (28). Note that now  $S_i$  is smaller than  $Q_c \propto \sqrt{1-\theta}$ , but has the same scale  $1-\theta$  as the consumed work  $W$ ; see our discussion after (32). Thus  $S_i$  cannot be neglected, and this explains why the Carnot efficiency is not reached in the equilibrium limit  $\theta \rightarrow 1$ ; see (11).

## VIII. CLASSICAL LIMIT.

We saw above that the optimization of the target quantity  $\chi = Q_c \zeta$  produced an inhomogeneous type of spectrum, where a batch of (quasi)degenerate energy levels is separated from the ground state by a gap. It is meaningful to carry out the optimization of  $\chi$  imposing a certain homogeneity in the spectra of  $\mathbf{H}$  and  $\mathbf{C}$ . The simplest situation of this type is the *equidistant* spectra

$$\varepsilon_n = (n-1)\varepsilon, \quad \mu_n = (n-1)\mu, \quad (54)$$

for  $\mathbf{H}$  and  $\mathbf{C}$ ; recall (3). For  $n \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ ,  $\mu \rightarrow 0$  these spectra correspond to the classical limit.

Thus, now we maximize  $\chi = Q_c \zeta$  imposing conditions (54). We found numerically that the optimal  $\mathcal{U}$  again corresponds to SWAP operation; see (20). For  $\chi = \chi(\bar{u}, \bar{v})$  we get

$$\chi = \frac{T_c \theta \ln^2 \frac{1}{\bar{v}}}{\ln \frac{1}{\bar{u}} - \theta \ln \frac{1}{\bar{v}}} \left[ \frac{\bar{v} - \bar{u}}{(1-\bar{v})(1-\bar{u})} - \frac{n(\bar{v}^n - \bar{u}^n)}{(1-\bar{v}^n)(1-\bar{u}^n)} \right],$$

where  $\bar{v} = e^{-\beta_c \varepsilon}$  and  $\bar{u} = e^{-\beta_h \mu}$  are found from maximizing  $\chi$ . The efficiency  $\zeta$  is still given by (15).

In the limit  $n \gg 1$  we get from maximizing  $\chi$ :

$$\bar{u} \rightarrow 1, \quad \bar{v} \rightarrow 1 \quad \text{and} \quad \frac{n(\bar{v}^n - \bar{u}^n)}{(1-\bar{v}^n)(1-\bar{u}^n)} \rightarrow 0, \quad (55)$$

implying that  $\chi$  and  $\zeta$  depend on one parameter  $\phi \equiv \frac{1-\bar{u}}{1-\bar{v}}$ :

$$\chi = \frac{T_c \theta (\phi - 1)}{\phi (\phi - \theta)}, \quad \zeta = \frac{\theta}{\phi - \theta}.$$

<sup>7</sup> In the definition (47) of  $\bar{p}$  one can as well employ  $T_h$  instead of  $T_c$ . This will not lead to serious changes, because we always assume that  $\theta = T_c/T_h$  is fixed.

The optimal value of this parameter is  $\phi = 1 + \sqrt{1 - \theta}$ . This leads to

$$\chi = \frac{T_c \theta}{(1 + \sqrt{1 - \theta})^2}, \quad \zeta = \zeta_{CA} = \frac{1}{\sqrt{1 - \theta}} - 1. \quad (56)$$

Thus for a large number of equidistant energy levels the maximization of  $\chi$  leads to homogeneity ( $\bar{\varepsilon} \rightarrow 0$ ,  $\bar{\mu} \rightarrow 0$ ), which is an indication of the classical limit. The efficiency  $\zeta$  in this constrained optimal situation is equal to  $\zeta_{CA}$ .

The above results refer to optimizing  $\chi$  in the limit  $n \gg 1$ . However, we confirmed numerically that the above values (56) for  $\chi$  and  $\zeta$ —obtained in the limit  $n \gg 1$ —are upper bounds for  $\chi$  and  $\zeta$  at a finite  $n$ .

Our conclusion is that the efficiency  $\zeta_{CA}$ —which is a lower bound for the efficiency during the unconstrained optimization of  $\chi$ —appears to be an upper bound for refrigerators that operate under equidistant (classical) spectra. The upper bound is reached in the limit  $n \gg 1$ .

These facts clarify to some extent why  $\zeta_{CA}$  was obtained as an upper bound for the efficiency within the finite-time thermodynamics (FTT) [21, 22]. Apparently, the quasi-equilibrium assumptions of FTT implied constraints which are equivalent to imposing the homogeneous spectra in our approach.

## IX. SUMMARY

We have studied a model of a refrigerator aiming to understand its optimal performance at a finite cooling power; see Fig. 1. The structure of the model is such that it can be optimized over almost all its parameters; additional constraints can and have been considered, though. We have confirmed the complementarity between optimizing the heat  $Q_c$  transferred from the cold bath  $T_c$  and efficiency  $\zeta$ : maximizing one nullifies the other. Similar effect for different models of quantum refrigerators are reported in [17–19].

To get a balance between  $Q_c$  and  $\zeta$  we have thus chosen to optimize their product  $\zeta Q_c$ . This leads to a *lower* bound  $\zeta_{CA} = \frac{1}{\sqrt{1 - \theta}} - 1$  ( $\theta \equiv \frac{T_c}{T_h}$ ) for the efficiency in

addition to the upper Carnot bound  $\zeta_C = \frac{1}{1 - \theta} - 1$ . The fact of  $\zeta > \zeta_{CA}$  implies that there is the lowest finite temperature reachable within one cycle of refrigeration; see (35).

The lower bound  $\zeta_{CA}$  is reached in the equilibrium limit  $T_c \rightarrow T_h$ . Constraining both systems to have homogeneous (classical) spectra,  $\zeta_{CA}$  is reached as an upper bound. This is just like within finite-time thermodynamics (FTT), when maximizing the product of the cooling-power and efficiency [21], or the ratio of the efficiency and the cycle time [22]. In this sense  $\zeta_{CA}$  seems to be universal. It may play the same role as the Curzon-Ahlborn efficiency for heat engines  $\eta_{CA}$ , which, again, is an upper bound within FTT [7], but appears as a lower bound for the engine models studied in [8]. For other opinions on the Curzon-Ahlborn efficiency for refrigerators see [23, 24].

The Carnot upper bound is asymptotically reached in the many-level limit of the model. We saw that this asymptotic convergence is related to a finite heat transferred per cycle, and we argued that it can also be related to a finite power if the relaxation scenario of the model refrigerator is chosen properly: provided that the cycle time is larger than the relaxation time, one can perform exponentially large number of refrigeration cycles before *inevitable* deviations from cyclicity—that in any case are there due to a finite cycle time—will accumulate. To our knowledge such an effect has never been seen so far for refrigerator models.

For the optimal refrigerator the transferred heat  $Q_c$  behaves as  $Q_c \propto T_c$  (in particular, for  $T_c \rightarrow 0$ ); see (13, 21, 41). This is in agreement with the optimal low-temperature behaviour of  $Q_c$  from the viewpoint of the third law [16, 20].

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- [1] G. Lindblad, *Non-Equilibrium Entropy and Irreversibility* (Reidel, Dordrecht, 1983).
  - [2] H.B. Callen, *Thermodynamics* (John Wiley, NY, 1985).
  - [3] L.D. Landau and E.M. Lifshitz, *Statistical Physics, I*, Pergamon Press Oxford, 1978.
  - [4] J. Gemmer, M. Michel and G. Mahler, *Quantum Thermodynamics* (Springer, 2004).
  - [5] V. Balzani, A. Credi and M. Venturi, *Molecular Devices and Machines* (Wiley-VCH, Weinheim, 2003).
  - [6] R.S. Berry, V.A. Kazakov, S. Sieniutycz, Z. Szwast, and A.M. Tsvilin, *Thermodynamic Optimization of Finite-Time Processes* (John Wiley & Sons, Chichester, 2000).
  - [7] F. Curzon and B. Ahlborn, Am. J. Phys. **43**, 22 (1975).
  - [8] A.E. Allahverdyan *et al.*, Phys. Rev. E **77**, 041118 (2008).
  - [9] Z.C. Tu, J. Phys. A **41**, 312003 (2008).
  - [10] Y. Izumida and K. Okuda, EPL **83**, 60003 (2008).
  - [11] T. Schmiedl and U. Seifert, EPL **81**, 20003 (2008).
  - [12] M. Esposito *et al.*, Phys. Rev. Lett. **102**, 130602 (2009).  
M. Esposito *et al.*, *Quantum-dot Carnot engine at maximum power*, arXiv:1001.2192.
  - [13] A.E. Allahverdyan, R. Balian and Th.M. Nieuwenhuizen, J. Mod. Opt. **51**, 2703 (2004).
  - [14] D. Janzing, J. Stat. Phys. **122**, 531 (2006).
  - [15] M.J. Henrich *et al.*, Europhys. Lett., **76**, 1057 (2006).
  - [16] R. Kosloff, E. Geva and J. M. Gordon, J. Appl. Phys. **87**, 8093 (2000).
  - [17] T. Feldman and R. Kosloff, Phys. Rev. E, **61**, 4774 (2000).

- [18] T. Feldmann and R. Kosloff, *Optimal Performance of Reciprocating Quantum Refrigerators*, arXiv:0906.0986.
- [19] D. Segal, Phys. Rev. Lett. **101**, 260601 (2008).
- [20] Y. Rezek *et al.*, EPL, **85**, 30008 (2009).
- [21] Z. Yan and J. Chen, J. Phys. D **23**, 136 (1990).
- [22] S. Velasco *et al.*, Phys. Rev. Lett. **78**, 3241 (1997).
- [23] B. Jimenez de Cisneros *et al.*, Phys. Rev. E **73**, 057103 (2006).
- [24] A. Calvo Hernandez *et al.*, Phys. Rev. E **63**, 037102 (2001).
- [25] R. Marathe, A. M. Jayannavar and A. Dhar, Phys. Rev. E **75**, 030103 (R) (2007).  
D. Segal and A. Nitzan, Phys. Rev. E **73**, 026109 (2006).  
J. Ren and B. Li, Phys. Rev. E **81**, 021111 (2010).
- [26] K. Sekimoto, F. Takagi and T. Hondou, Phys. Rev. E **62**, 7759 (2000).
- [27] K. Sekimoto and S. Sasa, J. Phys. Soc. Jpn. **66**, 3326 (1997).
- [28] A. E. Allahverdyan and Th. M. Nieuwenhuizen, Phys. Rev. E **71**, 046107 (2005).
- [29] R. B. Griffiths, J. Math. Phys. **6**, 1447 (1965).
- [30] M. O. Scully *et al.*, J. Mod. Opt. **49**, 2297 (2002).
- [31] A. Galindo and M. A. Martin-Delgado, Rev. Mod. Phys. **74**, 347 (2002).
- [32] N. Sangouard *et al.* Phys. Rev. A **72**, 062309 (2005).
- [33] M. J. Klein, in *Thermodynamics of Irreversible Processes, Scuola internazionale di fisica "Enrico Fermi"*, ed. by S. R. de Groot (Bologna, 1960).
- [34] A.W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*, (Academic Press, New York, 1979).
- [35] H. M. Partovi, Phys. Lett. A **137**, 440 (1989).
- [36] A. B. Brailovskii, V. L. Vaks, and V. V. Mityugov, Phys. Usp. **166**, 795 (1996).  
V. V. Mityugov, Phys. Usp. **170**, 681 (2000).
- [37] A.E. Allahverdyan *et al.*, Phys. Rev. E **71**, 046106 (2005).
- [38] R. Kubo, J. Phys. Soc. Jap. **12**, 570 (1957).  
P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).
- [39] M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, 1975).

## Appendix A: Collisional relaxation.

### 1. General consideration

The purpose of this discussion is to outline the general structure of a collisional relaxation process. Our presentation follows to [35, 36].

The thermal bath is modeled as a collection of  $N \gg 1$  independent equilibrium systems (particles) with initial density matrices  $\omega_i = \frac{1}{Z_i} \exp[-\beta H_i]$  and Hamiltonians  $H_i$ , where  $i = 1, \dots, N$ , and where  $1/\beta = T$  is the bath temperature. This formalizes the intuitive notion of the bath as a collection of many weakly-interacting particles.

The target system  $\mathbf{H}$  starts in [an arbitrary] initial state  $\rho_{\mathbf{H}}$  and has Hamiltonian  $H_{\mathbf{H}}$ . The collisional relaxation is realized when the particles of the bath sequentially interact [collide] with  $\mathbf{H}$ . Multiple collisions (between the target system and simultaneously two or

more bath particles) are neglected.

Consider the first collision. The initial state of  $\mathbf{H}$  and the first bath particle is  $\Omega_{1+\mathbf{H}} = \rho \otimes \omega_1$ . The interaction between them is realized via a unitary operator  $\mathcal{V}$ , so that the final state after the first collision is  $\Omega'_{1+\mathbf{H}} = \mathcal{V}\Omega_{1+\mathbf{H}}\mathcal{V}^\dagger$ . This unitary operator is generated by the full Hamiltonian  $H_{1+\mathbf{H}}$ :

$$H_{1+\mathbf{H}} = H_1 + H_{\mathbf{H}} + H_{1,\mathbf{H}}, \quad (\text{A1})$$

where  $H_{1,\mathbf{H}}$  is the interaction Hamiltonian. Define separate final states:

$$\rho' = \text{tr}_1 \Omega'_{1+\mathbf{H}}, \quad \omega'_1 = \text{tr}_{\mathbf{H}} \Omega'_{1+\mathbf{H}}, \quad (\text{A2})$$

where  $\text{tr}_1$  and  $\text{tr}_{\mathbf{H}}$  are the partial over the first particle and  $\mathbf{H}$ , respectively. Recall the definition (9) of the relative entropy. The unitarity of  $\mathcal{V}$  implies

$$S[\Omega'_{1+\mathbf{H}} || \rho' \otimes \omega_1] = \text{tr}[\Omega_{1+\mathbf{H}} \ln \Omega_{1+\mathbf{H}}] - \text{tr}[\Omega'_{1+\mathbf{H}} \ln(\rho' \otimes \omega_1)]. \quad (\text{A3})$$

Employing  $\omega_1 = \frac{1}{Z_1} \exp[-\beta H_1]$  and  $S[\Omega'_{1+\mathbf{H}} || \rho' \otimes \omega_1] \geq 0$  in (A3) we get

$$T\Delta S_{\mathbf{H}} + \Delta U_1 \geq 0, \quad (\text{A4})$$

where  $\Delta S_{\mathbf{H}} = \text{tr}[-\rho' \ln \rho' + \rho \ln \rho]$  and  $\Delta U_1 = \text{tr}(H_1 [\omega'_1 - \omega_1])$  are, respectively, the change of the entropy of  $\mathbf{H}$  and the average energy of the first particle.

We now require that the interaction  $\mathcal{V}$  conserves the average energy:

$$\Delta U_1 = -\Delta U_{\mathbf{R}}. \quad (\text{A5})$$

Using this in (A4) one has

$$\Delta U_{\mathbf{H}} - T\Delta S_{\mathbf{H}} \leq 0. \quad (\text{A6})$$

Since we did not use any special feature of the initial state of  $\mathbf{H}$ , (A6) holds for subsequent collisions of  $\mathbf{H}$  with the bath particles. Thus  $U_{\mathbf{H}} - TS_{\mathbf{H}}$  decays in time, and it should attain its minimum. It is well-known [35, 36] that this minimum is reached for the Gibbs matrix  $\rho \propto e^{-\beta H_{\mathbf{H}}}$ : collisions can drive  $\mathbf{H}$  to equilibrium starting from an arbitrary state [35, 36].

Condition (A5) expresses the average energy conservation. It is natural to use a more stringent condition according to which the sum of energies of  $\mathbf{H}$  and the bath particle 1 is conserved in time [36]:

$$[H_1 + H_{\mathbf{H}}, H_{1,\mathbf{H}}] = 0. \quad (\text{A7})$$

This condition makes the dynamics *autonomous*, since for any initial state of  $\mathbf{H} + 1$  the switching the interaction on and off does not cost energy and (A5) holds automatically.

For condition (A7) to be non-trivial, the operator  $H_1 + H_{\mathbf{H}}$  should have a degenerate spectrum. Otherwise due to  $[H_1 + H_{\mathbf{H}}, H_{1,\mathbf{H}}] = 0$  and (A7),  $H_{\mathbf{H}}$  and  $H_1$  will be

constants of motion, which means that no transfer of energy and thus no relaxation is possible.

Here are two crucial points of the collisional relaxation.

1. If the target system starts in the equilibrium state, this state does not change in time under subsequent collisions. This analogue of the zero law of thermodynamics is especially obvious from condition (A7), but it also holds simply from the conservation of the average energy (A5). Indeed, if  $\rho_{\mathbf{H}}$  is the equilibrium state,  $\rho_{\mathbf{H}} \propto e^{-\beta H_{\mathbf{H}}}$ , and also condition (A5) holds, the relative entropy  $S[\Omega'_{1+\mathbf{H}} || \rho_{\mathbf{H}} \otimes \omega_1]$  is equal to zero, which can happen only for  $\Omega'_{1+\mathbf{H}} = \rho_{\mathbf{H}} \otimes \omega_1$ .

2. No work is done for switching collisions on and off. This is clearly seen from (A7), which states that the free Hamiltonian is a constant of motion.

## 2. The relaxation time for a pertinent example

Now we study the relaxation time for an  $n$ -level system  $\mathbf{H}$  under collisional dynamics. We assume that  $n-1$  levels of  $\mathbf{H}$  coincide and have energy  $\varepsilon > 0$ . The lowest energy level is not degenerate and has energy zero. Importantly, we assume that condition (A7) holds meaning that the relaxation proceeds autonomously, i.e., without additional energy [work] costs on switching the interaction on and off. The initial (before colliding with the first bath particle) density matrix of  $\mathbf{H}$  is assumed to be Gibbsian at temperature  $T_0 = 1/\beta_0$ :

$$\rho = \frac{e^{-\beta_0 H_{\mathbf{H}}}}{Z_{\mathbf{H}}(\beta_0)} = r \mathbf{P}_0 + \frac{1-r}{n-1} \mathbf{P}_{\varepsilon}, \quad r = \frac{1}{1 + (n-1)e^{-\beta_0 \varepsilon}},$$

where  $\mathbf{P}_0 = |0\rangle\langle 0|$  and  $\mathbf{P}_{\varepsilon}$  is the projectors on the  $n-1$ -dimensional eigen-space of  $\rho$  with eigenvalue  $\varepsilon$ .

To satisfy the degeneracy of the interaction Hamiltonian [see our discussion after (A7)] we assume that the first bath particle has (among others energies) energy levels  $E$  and  $E+\varepsilon$ . The degeneracies of these levels are  $n_E^{[1]}$  and  $n_{E+\varepsilon}^{[1]}$ , respectively. The equilibrium density matrix of the bath particle 1 is written as

$$\rho_1 = \tilde{\rho}_1 + r_E^{[1]} \mathbf{P}_E^{[1]} + r_{\varepsilon+E}^{[1]} \mathbf{P}_{\varepsilon+E}^{[1]}, \quad (\text{A8})$$

$$r_{E_{\alpha}}^{[1]} = e^{-\beta E_{\alpha}} / Z_1, \quad Z_1 = \sum_{\alpha} n_{E_{\alpha}}^{[1]} e^{-\beta E_{\alpha}}, \quad (\text{A9})$$

where  $r_E^{[1]}$  and  $r_{E+\varepsilon}^{[1]}$  are the Boltzmann weights for the energy levels  $E$  and  $E+\varepsilon$ , respectively, the summation in (A9) is taken over all energy levels of the bath particle,  $\mathbf{P}_E^{[1]}$  and  $\mathbf{P}_{\varepsilon+E}^{[1]}$  are the projectors on the corresponding sub-spaces,

$$\text{tr} \mathbf{P}_E^{[1]} = n_E^{[1]}, \quad \text{tr} \mathbf{P}_{\varepsilon+E}^{[1]} = n_{E+\varepsilon}^{[1]}, \quad (\text{A10})$$

and where  $\tilde{\rho}_1$  in (A8) is the remainder of  $\rho_1$ .

It is assumed that the unitary operator  $\mathcal{V}$  responsible for the interaction operates within the sub-space with

the projector  $\mathbf{P}_{\varepsilon} \otimes \mathbf{P}_E^{[1]} + \mathbf{P}_0 \otimes \mathbf{P}_{E+\varepsilon}^{[1]}$  (this sub-space has energy  $E+\varepsilon$ ), i.e.,

$$[\mathcal{V}, \mathbf{P}_{\varepsilon} \otimes \mathbf{P}_E^{[1]} + \mathbf{P}_0 \otimes \mathbf{P}_{E+\varepsilon}^{[1]}] = 0. \quad (\text{A11})$$

Then the post-collision density matrix  $\rho'$  of  $\mathbf{H}$  reads

$$\rho' = \text{tr}_1 \mathcal{V} \rho \otimes \rho_1 \mathcal{V}^{\dagger} = \rho - \left( r r_{E+\varepsilon}^{[1]} - r_E^{[1]} \frac{1-r}{n-1} \right) [n_{E+\varepsilon}^{[1]} \mathbf{P}_0 - \text{tr}_1 \mathcal{V} \mathbf{P}_0 \otimes \mathbf{P}_{E+\varepsilon}^{[1]} \mathcal{V}^{\dagger}]$$

Clearly,  $\rho'$  commutes with  $H_{\mathbf{H}}$ . For simplicity, we choose  $\mathcal{V}$  such that the degeneracy of  $\rho$  is not resolved, i.e., in the state  $\rho'$ , the occupations of the higher energy levels of  $\mathbf{H}$  are equal. This means we need to keep track of the lowest energy-level occupation  $\langle 0 | \rho' | 0 \rangle \equiv r'$  only:

$$r' - r = -A [r - r_{\text{eq}}], \quad r_{\text{eq}} \equiv \frac{1}{1 + (n-1)e^{-\beta \varepsilon}}, \quad (\text{A12})$$

$$A \equiv \frac{r_E^{[1]}}{r_{\text{eq}}(n-1)} \left[ n_{E+\varepsilon}^{[1]} - \langle 0 | (\text{tr}_1 \mathcal{V} \mathbf{P}_0 \otimes \mathbf{P}_{E+\varepsilon}^{[1]} \mathcal{V}^{\dagger}) | 0 \rangle \right], \quad (\text{A13})$$

where  $r_{\text{eq}}$  is the equilibrium value of  $r$ .  $A$  can be maximized over the unitary  $\mathcal{V}$  [under condition (A11)]

$$A_{\text{max}} = \frac{r_E^{[1]} \min [n_{E+\varepsilon}^{[1]}, n_E^{[1]}(n-1)]}{r_{\text{eq}}(n-1)}. \quad (\text{A14})$$

Using (A9) one can show that  $A \leq A_{\text{max}} \leq 1$ : after first collision  $\mathbf{H}$  gets closer to its equilibrium state; see (A12). This equation obviously generalizes to subsequent collisions [we revert from (A15) to (A12) for  $m=1$ ]:

$$r'_{[m]} - r'_{[m-1]} = (1-A)^m [r - r_{\text{eq}}], \quad (\text{A15})$$

It is seen that (A15) predicts exponential (with respect to the number of collisions) relaxation towards the equilibrium value  $r_{\text{eq}}$  of  $r$ . The approach to equilibrium is governed by the factor  $(1-A)^n$  meaning that when  $|A| \ll 1$  the effective number of collisions after which the equilibrium is established equals to  $-1/[\ln(1-A)]$ .

Now the shortest relaxation corresponds to just one collision and it is reached for  $A=1$ , e.g.,  $r_E^{[1]} = r_{\text{eq}}$  and  $n_{E+\varepsilon}^{[1]} = n-1$  in (A14). Then the corresponding unitary operator  $\mathcal{V}$  is the SWAP operation. The relaxation time in this case amounts to one inter-collision time.

It should be clear that there is no upper limit on the relaxation time. The latter can be arbitrary large, e.g., due to  $\mathcal{V}$  converging to 1 in (A13). Various intermediate cases can be studied with help of (A14). In particular, it is not difficult to identify regimes, where the relaxation time scales as  $\propto \ln n$ .